Introduction to Probabilistic Graphical Models

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Schedule

Refresher of Probabilities
Introduction to Probabilistic Graphical Models
Probabilistic Inference
Learning Conditional Random Fields
MAP Prediction / Energy Minimization
Learning Structured Support Vector Machines

Links to slide download: `http://pub.ist.ac.at/~chl/courses/PGM_W16/`

Password for ZIP files (if any): `pgm2016`

Email for questions, suggestions or typos that you found: `chl@ist.ac.at`

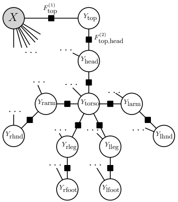
Supervised Learning Problem

- Given training examples $(x^1, y^1), \ldots, (x^N, y^N) \in \mathcal{X} \times \mathcal{Y}$
  $x \in \mathcal{X}$: input, *e.g.* image
  $y \in \mathcal{Y}$: structured output, *e.g.* human pose, sentence

Images: HumanEva dataset

Goal: be able to make predictions for new inputs, i.e. learn a function $f : \mathcal{X} \to \mathcal{Y}$.

Supervised Learning Problem



Step 1: define a proper graph structure of X and Y

Step 2: define a proper parameterization of $p(y|x;\theta)$

$$p(y|x;\theta) = \frac{1}{Z} e^{\sum_{i=1}^{d} \theta_i \phi_i(x,y)}$$

Step 3: learn parameters θ^* from training data

e.g. maximum likelihood

Step 4: for new $x \in \mathcal{X}$, make prediction

e.g. $y^* = \operatorname{argmax}_{y \in \mathcal{Y}} p(y|x;\theta^*)$

($\rightarrow$ today)

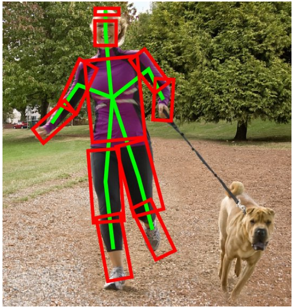
MAP Prediction / Energy Minimization

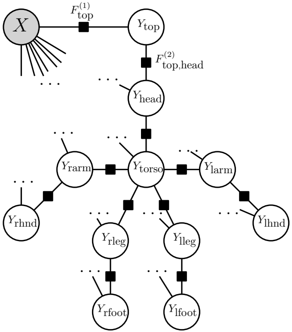
$\operatorname{argmax}_y p(y|x)$ / $\operatorname{argmin}_y E(y, x)$

MAP Prediction / Energy Minimization

- Exact Energy Minimization
  - Belief Propagation on chains/trees
  - Graph-Cuts for submodular energies
  - Integer Linear Programming
- Approximate Energy Minimization
  - Linear Programming Relaxations
  - Local Search Methods
    - Iterative Conditional Modes
    - Multi-label Graph Cuts
  - Simulated Annealing

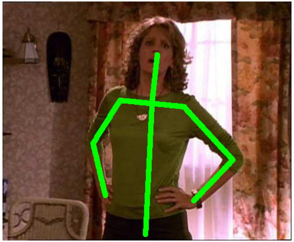
Example: Pictorial Structures / Deformable Parts Model





- **Tree-structured model** for articulated pose (Felzenszwalb and Huttenlocher, 2000), (Fischler and Elschlager, 1973), (Yang and Ramanan, 2013), (Pishchulin *et al.*, 2012)

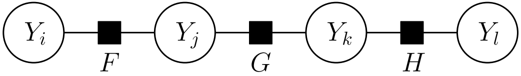
Example: Pictorial Structures / Deformable Parts Model



- most likely configuration $y^* = \underset{y \in \mathcal{Y}}{\operatorname{argmax}}\, p(y|x) = \underset{y}{\operatorname{argmin}}\, E(y, x)$

Energy Minimization – Belief Propagation

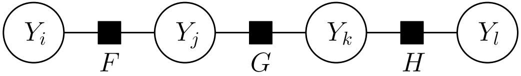
Chain model: same trick as for *inference*: **belief propagation**



$$\min_y E(y) = \min_{y_i,y_j,y_k,y_l} \quad E_F(y_i, y_j) + E_G(y_j, y_k) + E_H(y_k, y_l)$$

Energy Minimization – Belief Propagation

Chain model: same trick as for *inference*: **belief propagation**



$$\begin{aligned}\min_{y} E(y) &= \min_{y_i, y_j, y_k, y_l} \quad E_F(y_i, y_j) + E_G(y_j, y_k) + E_H(y_k, y_l) \\ &= \min_{y_i, y_j} \left[E_F(y_i, y_j) + \min_{y_k} \left[E_G(y_j, y_k) + \min_{y_l} E_H(y_k, y_l) \right] \right]\end{aligned}$$

Energy Minimization – Belief Propagation

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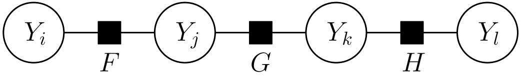

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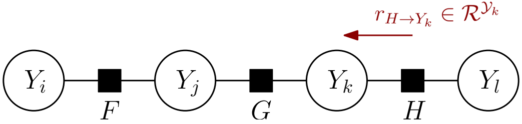
$$= \min_{y_i,y_j} \Big[E_F(y_i,y_j) + \min_{y_k} \big[E_G(y_j,y_k) + \underbrace{\min_{y_l} E_H(y_k,y_l)}_{r_{H\to Y_k}(y_k)} \big]\Big]$$

Energy Minimization – Belief Propagation

Chain model: same trick as for *inference*: **belief propagation**



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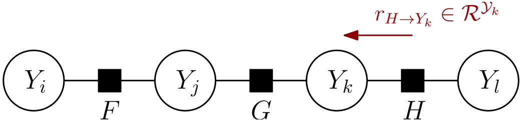
Energy Minimization – Belief Propagation

Chain model: same trick as for *inference*: **belief propagation**



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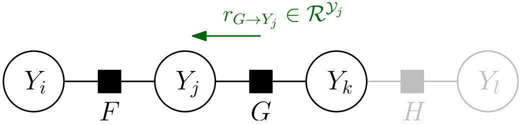
Energy Minimization – Belief Propagation

Chain model: same trick as for *inference*: **belief propagation**



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$$= \min_{y_i, y_j} \left[E_F(y_i, y_j) + r_{G \to Y_j}(y_j) \right] \quad \ldots$$

Energy Minimization – Belief Propagation

Chain model: same trick as for *inference*: **belief propagation**

$$\min_y E(y) = \min_{y_i,y_j,y_k,y_l} \quad E_F(y_i,y_j) + E_G(y_j,y_k) + E_H(y_k,y_l)$$

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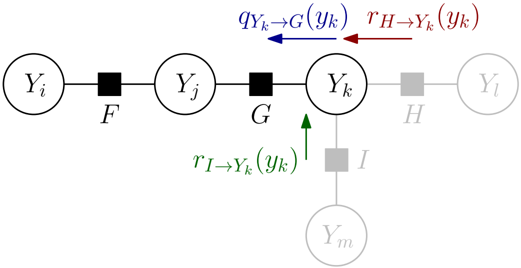
$$= \min_{y_i,y_j}\Big[E_F(y_i,y_j) + r_{G\to Y_j}(y_j)\Big] \quad \ldots$$

- actual argmax by backtracking which choices were maximal

Energy Minimization – Belief Propagation

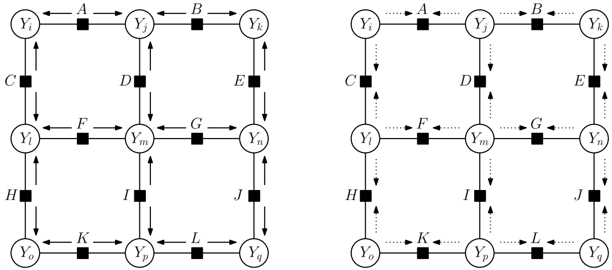
Tree models:



- $q_{H \to Y_k}(y_k) = \min_{y_l} E_H(y_k, y_l)$
- $q_{I \to Y_k}(y_k) = \min_{y_m} E_I(y_k, y_m)$
- $q_{Y_k \to G}(y_k) = q_{H \to Y_k}(y_k) + q_{I \to Y_k}(y_k)$

min-sum (more commonly **max-sum**) **belief propagation**

Belief Propagation in Cyclic Graphs



Loopy Max-Sum Belief Propagation

Same problem as in probabilistic inference:

- no guarantee of convergence
- no guarantee of optimality

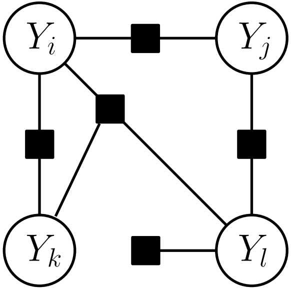
Some convergent variants exist, *e.g.* TRW-S [Kolmogorov, PAMI 2006]

Cyclic Graphs

In general, MAP prediction/energy minimization in models with cycles or higher-order terms is **intractable** (NP-hard).

Some important exceptions:

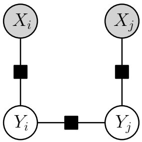
- low tree-width [Lauritzen, Spiegelhalter, 1988]
- **binary states, pairwise submodular interactions [Boykov, Jolly, 2001]**
- binary states, only pairwise interactions, planar graph [Globerson, Jaakkola, 2006]
- special (Potts $\mathcal{P}^n$) higher order factors [Kohli, Kumar, 2007]
- perfect graph structure [Jebara, 2009]



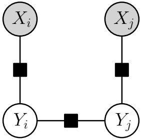
Submodular Energy Functions

- Binary variables: $\mathcal{Y}_i = \{0, 1\}$ for all $i \in \mathcal{V}$
- Energy function: unary and pairwise factors

$$E(y; x, w) = \sum_{i \in \mathcal{V}} E_i(y_i) + \sum_{(i,j) \in \mathcal{E}} E_{ij}(y_i, y_j)$$



Submodular Energy Functions



- Binary variables: $\mathcal{Y}_i = \{0, 1\}$ for all $i \in \mathcal{V}$
- Energy function: unary and pairwise factors

$$E(y; x, w) = \sum_{i \in \mathcal{V}} E_i(y_i) + \sum_{(i,j) \in \mathcal{E}} E_{ij}(y_i, y_j)$$

- Restriction 1 (without loss of generality):

$$E_i(y_i) \geq 0$$

(always achievable by adding a constant to E)

- Restriction 2 (**submodularity**):

$$\begin{aligned} E_{ij}(y_i, y_j) &= 0, && \text{if } y_i = y_j, \\ E_{ij}(y_i, y_j) &= E_{ij}(y_j, y_i) \geq 0, && \text{otherwise.} \end{aligned}$$

"neighbors prefer to have the same labels"

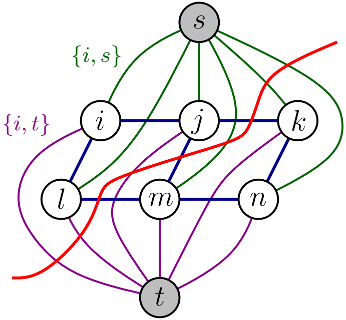
Graph-Cuts Algorithm for Submodular Energy Minimization [Greig *et al.*, 1989]

If conditions are fulfilled, energy minimization can be performed by a solving s-t-**mincut**:

- construct auxiliary undirected graph
- one node $\{i\}_{i \in V}$ per variable
- two extra nodes: source s, sink t
- weighted edges

Edge	weight
$\{i, j\}$	$E_{ij}(y_i = 0, y_j = 1)$
$\{i, s\}$	$E_i(y_i = 1)$
$\{i, t\}$	$E_i(y_i = 0)$

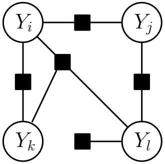
- find s-t-cut of minimal weight (polynomial time using max-flow theorem)



From minimal weight cut we recover labeling of minimal energy:

- $y_i^* = 1$ if edge $\{i, s\}$ is cut. Otherwise $y_i^* = 0$

Integer Linear Programming (ILP)



General energy $E(y) = \sum_F E_F(y_F)$

- variables with more than 2 states
- higher-order factors (more than 2 variables)
- non-submodular factors

Formulate as integer linear program (ILP)

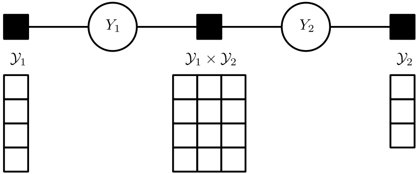
- linear objective function
- linear constraints
- variables to optimize over are integer-valued

ILPs are in general NP-hard, but some individual instances can be solved

- standard optimization toolboxes: *e.g.* CPLEX, Gurobi, COIN-OR, ...

Integer Linear Programming (ILP)

Example:

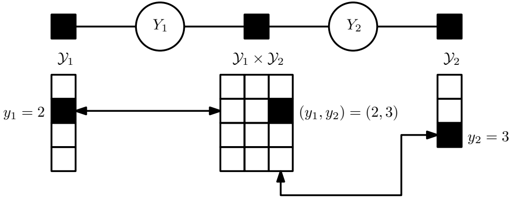


Encode assignment in indicator variables:

- $z_1 \in \{0,1\}^{\mathcal{Y}_1}$ $\qquad z_{1;k} = 1 \;\Leftrightarrow\; [\![y_1 = k]\!]$
- $z_2 \in \{0,1\}^{\mathcal{Y}_2}$ $\qquad z_{2;l} = 1 \;\Leftrightarrow\; [\![y_2 = l]\!]$
- $z_{12} \in \{0,1\}^{\mathcal{Y}_1 \times \mathcal{Y}_2}$ $\qquad z_{12;kl} = 1 \;\Leftrightarrow\; [\![y_1 = k \wedge y_2 = l]\!]$

Integer Linear Programming (ILP)

Example:



Encode assignment in indicator variables:

- $z_1 \in \{0,1\}^{\mathcal{Y}_1}, \qquad z_2 \in \{0,1\}^{\mathcal{Y}_2}, \qquad z_{12} \in \{0,1\}^{\mathcal{Y}_1 \times \mathcal{Y}_2}$

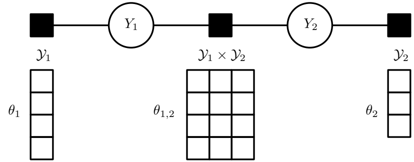
Consistency Constraints:

$$\sum_{k \in \mathcal{Y}_1} z_{1;k} = 1, \quad \sum_{l \in \mathcal{Y}_2} z_{2;l} = 1, \quad \sum_{k,l \in \mathcal{Y}_1 \times \mathcal{Y}_2} z_{12;kl} = 1 \quad \text{(indicator property)}$$

$$\sum_{k \in \mathcal{Y}_1} z_{12;kl} = z_{2;l} \qquad \sum_{l \in \mathcal{Y}_2} z_{12;kl} = z_{1;k} \qquad \text{(consistency)}$$

Integer Linear Programming (ILP)

Example: $E(y_1, y_2) = E_1(y_1) + E_{12}(y_1, y_2) + E_2(y_2)$



Define coefficient vectors:

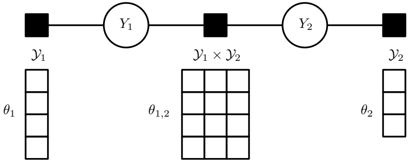
- $\theta_1 \in \mathbb{R}^{\mathcal{Y}_1}$ $\theta_{1;k} = E_1(k)$
- $\theta_2 \in \mathbb{R}^{\mathcal{Y}_2}$ $\theta_{2;l} = E_2(l)$
- $\theta_{12} \in \mathbb{R}^{\mathcal{Y}_1 \times \mathcal{Y}_2}$ $\theta_{12;kl} = E_{1,2}(k, l)$

Energy is a linear function of unknown z:

$$E(y_1, y_2) = \sum_{i \in V} \sum_{k \in \mathcal{Y}_i} \theta_{i;k} [\![y_i = k]\!] + \sum_{i,j \in \mathcal{E}} \sum_{k,l \in \mathcal{Y}_i \times \mathcal{Y}_j} \theta_{ij;kl} [\![y_i = k \wedge y_j = l]\!]$$

Integer Linear Programming (ILP)

Example: $E(y_1, y_2) = E_1(y_1) + E_{12}(y_1, y_2) + E_2(y_2)$



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Energy is a linear function of unknown z:

$$E(y_1, y_2) = \sum_{i \in V} \sum_{k \in \mathcal{Y}_i} \theta_{i;k} z_{i;k} + \sum_{(i,j) \in \mathcal{E}} \sum_{(k,l) \in \mathcal{Y}_i \times \mathcal{Y}_j} \theta_{ij;kl} z_{ij;kl}$$

Integer Linear Programming (ILP)

$$\min_{\mathbf{z}} \sum_{i \in V} \sum_{k \in \mathcal{Y}_i} \theta_{i;k} z_{i;k} + \sum_{(i,j) \in \mathcal{E}} \sum_{(k,l) \in \mathcal{Y}_i \times \mathcal{Y}_j} \theta_{ij;kl} z_{ij;kl}$$

subject to

$$z_{i;k} \in \{0,1\} \quad \text{for all } i \in V, \ \forall k \in \mathcal{Y}_i,$$

$$z_{ij;kl} \in \{0,1\} \quad \text{for all } (i,j) \in \mathcal{E}, (k,l) \in \mathcal{Y}_i \times \mathcal{Y}_j,$$

$$\sum_{k \in \mathcal{Y}_i} z_{i;k} = 1, \quad \text{for all } i \in V,$$

$$\sum_{k,l \in \mathcal{Y}_i \times \mathcal{Y}_j} z_{ij;kl} = 1, \quad \text{for all } (i,j) \in \mathcal{E},$$

$$\sum_{k \in \mathcal{Y}_i} z_{ij;kl} = z_{j;l} \quad \text{for all } (i,j) \in \mathcal{E}, \ l \in \mathcal{Y}_j,$$

$$\sum_{l \in \mathcal{Y}_j} z_{ij;kl} = z_{i;k} \quad \text{for all } (i,j) \in \mathcal{E}, \ k \in \mathcal{Y}_i,$$

Integer Linear Programming (ILP)

$$\min_{\mathbf{z}} \quad \sum_{i \in V} \sum_{k \in \mathcal{Y}_i} \theta_{i;k} z_{i;k} + \sum_{(i,j) \in \mathcal{E}} \sum_{(k,l) \in \mathcal{Y}_i \times \mathcal{Y}_j} \theta_{ij;kl} z_{ij;kl}$$

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$$\sum_{l \in \mathcal{Y}_j} z_{ij;kl} = z_{i;k} \quad \text{for all } (i,j) \in \mathcal{E},\ k \in \mathcal{Y}_i,$$

NP-hard to solve because of integrality constraints.

Linear Programming (LP) Relaxation

$$\min_{\mathbf{z}} \sum_{i \in V} \sum_{k \in \mathcal{Y}_i} \theta_{i;k} z_{i;k} + \sum_{(i,j) \in \mathcal{E}} \sum_{(k,l) \in \mathcal{Y}_i \times \mathcal{Y}_j} \theta_{ij;kl} z_{ij;kl}$$

subject to

$$\cancel{z_{i;k} \in \{0,1\}} \quad z_{i;k} \in [0,1] \quad \text{for all } i \in V,\ \forall k \in \mathcal{Y}_i,$$

$$\cancel{z_{ij;kl} \in \{0,1\}} \quad z_{ij;kl} \in [0,1] \quad \text{for all } (i,j) \in \mathcal{E}, (k,l) \in \mathcal{Y}_i \times \mathcal{Y}_j,$$

$$\sum_{k \in \mathcal{Y}_i} z_{i;k} = 1, \quad \text{for all } i \in V,$$

$$\sum_{k,l \in \mathcal{Y}_i \times \mathcal{Y}_j} z_{ij;kl} = 1, \quad \text{for all } (i,j) \in \mathcal{E},$$

$$\sum_{k \in \mathcal{Y}_i} z_{ij;kl} = z_{j;l} \quad \text{for all } (i,j) \in \mathcal{E},\ l \in \mathcal{Y}_j,$$

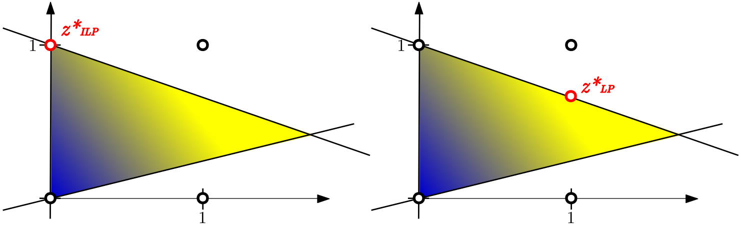
$$\sum_{l \in \mathcal{Y}_j} z_{ij;kl} = z_{i;k} \quad \text{for all } (i,j) \in \mathcal{E},\ k \in \mathcal{Y}_i,$$

Relax constraints $\rightarrow$ optimization problem becomes tractable

Linear Programming (LP) Relaxation

Solution $\mathbf{z}^*_{LP}$ might have fractional values

- $\rightarrow$ no corresponding labeling $y \in \mathcal{Y}$
- $\rightarrow$ round LP solution to $\{0, 1\}$ values



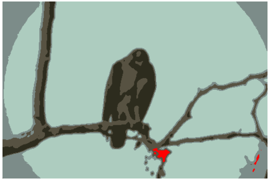
Problem:

- rounded solution usually not optimal, *i.e.* not identical to ILP solution

LP relaxations perform approximate energy minimization

Linear Programming (LP) Relaxation

Example: color quantization



Example: stereo reconstruction

Images: Berkeley Segmentation Dataset

Local Search

Avoid getting fractional solutions: energy minimization by **local search**

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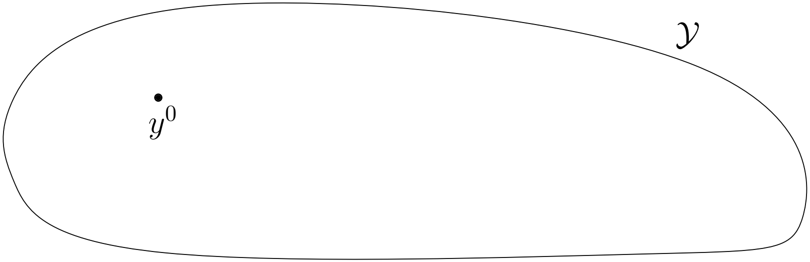

- choose starting labeling y^0

Local Search

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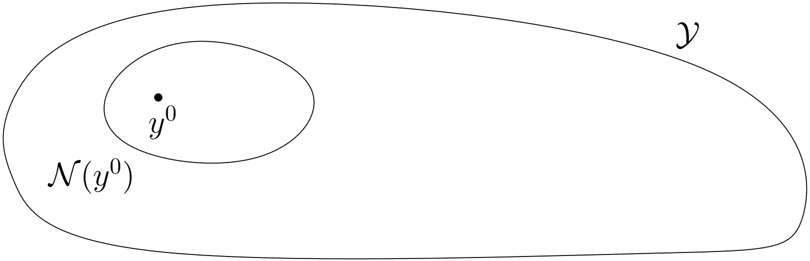

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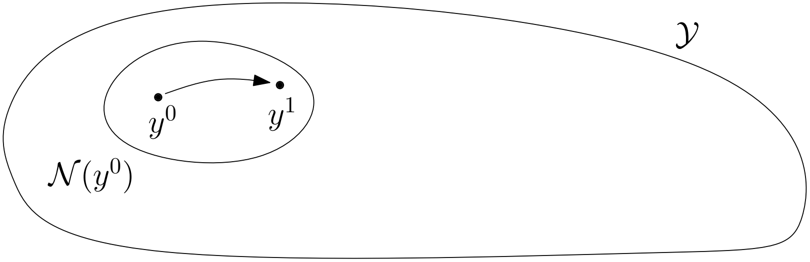

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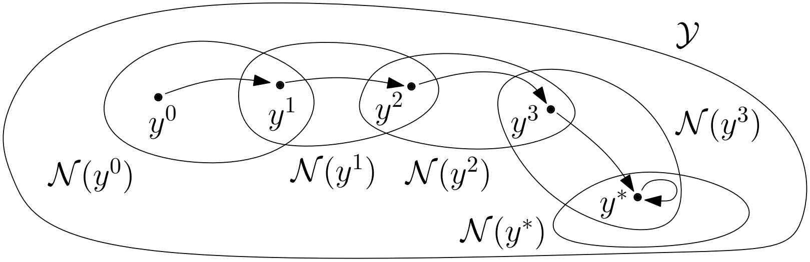

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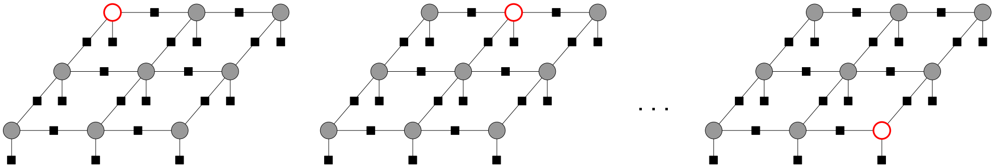
- iterate until no more changes

Iterated Conditional Modes (ICM) [Besag, 1986]

Define local neighborhoods:

- $\mathcal{N}_i(y) = \{(y_1, \dots, y_{i-1}, \bar{y}, y_{i+1}, \dots, y_n) | \bar{y} \in \mathcal{Y}_i\}$ for $i \in V$.

  all labeling reachable from y by changing value of y_i



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ICM procedure:

- neighborhood $\mathcal{N}(y) = \bigcup_{i \in V} \mathcal{N}_i(y)$

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Observation: larger neighborhood sizes are better

- ICM: $|\mathcal{N}(y)|$ linear in $|V|$

  $\rightarrow$ many iterations to explore exponentially large $\mathcal{Y}$
- ideal: $|\mathcal{N}(y)|$ exponential in $|V|$,

  $\rightarrow$ but: we must ensure that $\operatorname{argmin}_{y \in \mathcal{N}(y)} E(y)$ remains tractable

Multilabel Graph-Cut: α-expansion

- $E(y)$ with unary and pairwise terms
- $\mathcal{Y}_i = \mathcal{L} = \{1, \ldots, K\}$ for $i \in V$ (multi-class)

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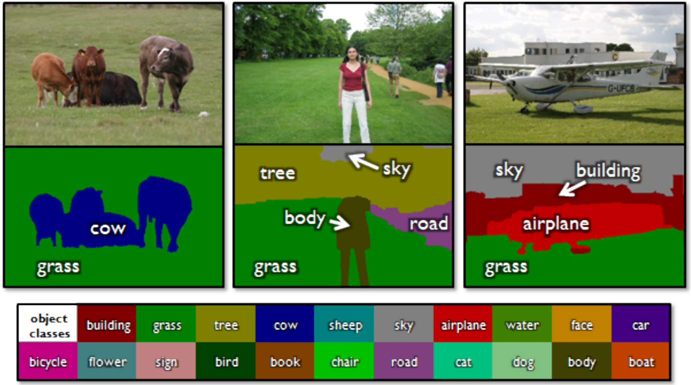
Example: semantic segmentation



Multilabel Graph-Cut: α-expansion

- $E(y)$ with unary and pairwise terms
- $\mathcal{Y}_i = \mathcal{L} = \{1, \dots, K\}$ for $i \in V$ (multi-class)

Algorithm

- initialize y^0 arbitrarily (*e.g.* everything label 0)
- repeat
  - for any $\alpha \in \mathcal{L}$
    - construct neighborhood:

$$\mathcal{N}(y) = \big\{(\bar{y}_1, \dots, \bar{y}_{|V|}) \ : \ \bar{y}_i \in \{y_i, \alpha\}\big\}$$

"each variable can keep its value or switch to α"

    - solve $y \leftarrow \operatorname{argmin}_{y \in \mathcal{N}(y)} E(y)$
- until y has not changed for a whole iteration

Multilabel Graph-Cut: α-expansion

Theorem **[Boykov et al. 2001]**
If all pairwise terms are *metric*, *i.e.* for all $(i,j) \in \mathcal{E}$

$$E_{ij}(k,l) \geq 0 \quad \text{with} \quad E_{ij}(k,l) = 0 \Leftrightarrow k = l$$

$$E_{ij}(k,l) = E_{ij}(l,k)$$

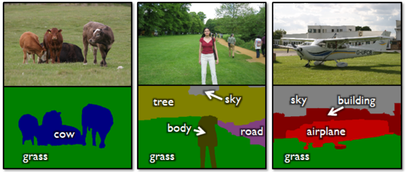
$$E_{ij}(k,l) \leq E_{ij}(k,m) + E_{ij}(m,l) \quad \text{for all } k,l,m$$

Then $\operatorname{argmin}_{y \in \mathcal{N}(y)} E(y)$ can be solved optimally using GraphCut.

Theorem **[Veksler 2001].** The solution, y_α, returned by α-expansion fulfills

$$E(y_\alpha) \;\leq\; 2c \cdot \min_{y \in \mathcal{Y}} E(y) \quad \text{for } c = \max_{(i,j)\in\mathcal{E}} \frac{\max_{k \neq l} E_{ij}(k,l)}{\min_{k \neq l} E_{ij}(k,l)}$$

Example: Semantic Segmentation



$$E(y) = \sum_{i \in V} E_i(y_i) + \lambda \sum_{(i,j) \in \mathcal{E}} [\![y_i \neq y_j]\!]$$ "Potts model"

- $E_{ij}(k,l) \geq 0 \qquad E_{ij}(k,l) = 0 \Leftrightarrow k = l \qquad E_{ij}(k,l) = E_{ij}(l,k) \qquad \checkmark$
- $E_{ij}(k,l) \leq E_{ij}(k,m) + E_{ij}(m,l) \qquad \checkmark$
- $c = \max_{(i,j)\in\mathcal{E}} \frac{\max_{k\neq l} E_{ij}(k,l)}{\min_{k\neq l} E_{ij}(k,l)} = 1$
- factor-2 approximation guarantee: $E(y_\alpha) \leq 2 \min_{y \in \mathcal{Y}} E(y)$

Example: Stereo Estimation

$$E(y) = \sum_{i \in V} E_i(y_i) + \lambda \sum_{(i,j) \in \mathcal{E}} |y_i - y_j|$$

- $|y_i - y_j|$ is metric $\checkmark$
- $c = \max_{(i,j) \in \mathcal{E}} \frac{\max_{k \neq l} E_{ij}(k,l)}{\min_{k \neq l} E_{ij}(k,l)} = |\mathcal{L} - 1|$
- weak guarantees, but often close to optimal labelings in practice

Images: Middlebury stereo vision dataset

Sampling

Sampling was a general purpose probabilistic inference method. Can we use it for prediction?

MAP prediction from samples:

- $S = \{x^1, \dots, x^N\}$ samples from $p(x)$
- $\hat{x}^* \leftarrow \operatorname{argmax}_{x \in S} p(x)$
- **output** $\hat{x}^*$

Problem:

- will need many samples
- with $x^* = \operatorname{argmax}_{x \in \mathcal{X}} p(x)$: $\Pr(\hat{x}^* \neq x^*) = (1 - p(x^*))^N \approx 1 - Np(x^*)$
- for graphical model, probability values are tiny, e.g. $p(x^*) = 10^{-100}$ can easily happen
- $N \approx 5 \cdot 10^{99}$ required to have 50% chance

Sampling

Let's construct a better distribution:

Idea 2: Form a new distribution, p', that has all its probability mass at the location of maximum of p

$$p'(x) = [\![x = x^*]\!] \qquad \text{for } x^* = \operatorname*{argmax}_{x \in \mathcal{X}} p(x)$$

and sample from it

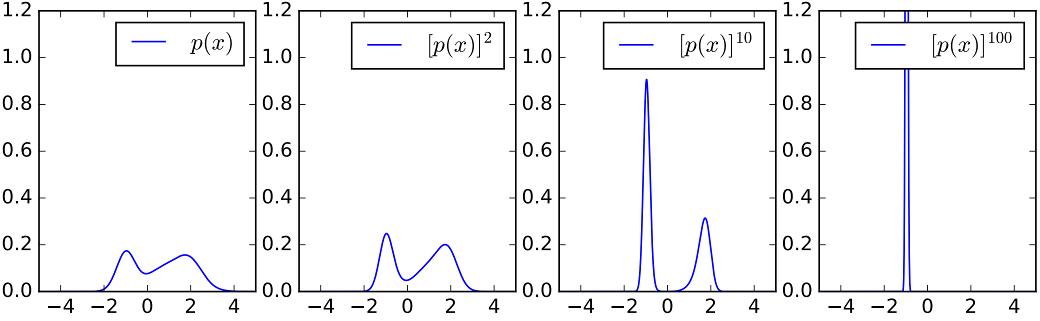
Advantage: we need only 1 sample

Problem: to define p' we need x^*, which is what we're after.

Sampling

Idea 3: do the same as idea 2, but more implicitly:

$$p_\beta(x) \propto [p(x)]^\beta \qquad \text{for very large } \beta$$



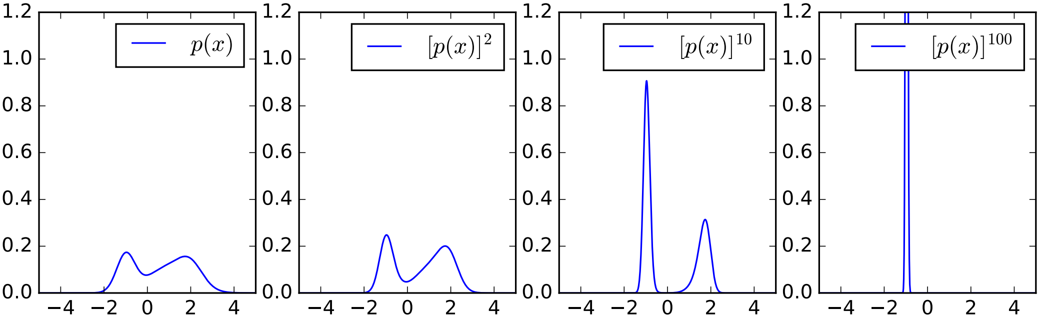
Particularly easy for distributions in exponential form: $p(x) \propto e^{-E(x)}$ becomes $p_\beta(x) \propto e^{-\beta E(x)}$

Simulated Annealing

Practical questions: How to choose β? How to sample from p_β?

These are often coupled, especially sampling works by a Monte Carlo Markov Chain (MCMC):

- samples $x^1, x^2, \dots$ from MCMC are dependent,
- two consecutive samples are often similar to each other $\rightarrow$ random walk,
- for a distribution with multiple peaks that are separated by a low-probability region, MCMC sampling will jump around a peak, but very rarely switch peaks



Sampling

Image: By Kingpin13 - Own work, CC0, https://commons.wikimedia.org/w/index.php?curid=25010763

MAP Prediction / Energy Minimization – Summary

Task: compute $\operatorname{argmin}_{y \in \mathcal{Y}} E(y|x)$

Exact Energy Minimization

Only possible for certain models:

- trees/forests: max-sum belief propagation
- general graphs: junction chain algorithm (if tractable)
- submodular energies: GraphCut
- general graphs: integer linear programming (if tractable)

Approximate Energy Minimization

Many techniques with different properties and guarantees:

- linear programs relaxations, ICM, α-expansion

Best choices depends on model and requirements.

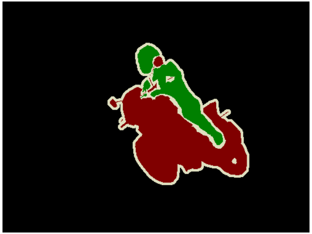
Loss functions

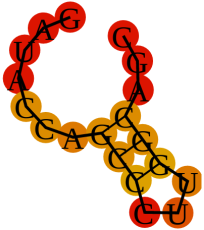
Loss functions

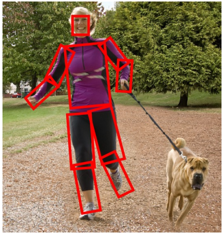
We model structured data, *e.g.* $y = (y_1, \ldots, y_m)$. What makes a good prediction?

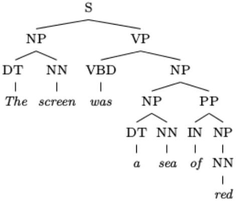
- The *loss function* is application dependent

$$\Delta : \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}_+,$$









Example 1: 0/1 loss

Loss is 0 for perfect prediction, 1 otherwise:

$$\Delta_{0/1}(\bar{y}, y) = [\![\bar{y} \neq y]\!] = \begin{cases} 0 & \text{if } \bar{y} = y \\ 1 & \text{otherwise} \end{cases}$$

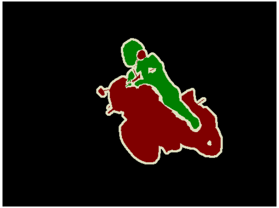
Every mistake is equally bad. Rarely very useful for *structured data*, *e.g.*

- handwriting recognition: one letter wrong is as bad as all letters wrong
- image segmentation: one pixel wrong is as bad as all pixels wrong
- automatic translation: a missing article is as bad as completely random output

Example 2: Hamming loss

Count the number of mislabeled variables:

$$\Delta_H(\bar{y}, y) = \frac{1}{m} \sum_{i=1}^{m} [\![\bar{y}_m \neq y_m]\!]$$



x:	I	need	a	coffee	break
$\bar{y}$:	subject	verb	article	object	object
y:	subject	verb	article	object	verb
$[\![\bar{y}_m \neq y_m]\!]$	0	0	0	0	1

$\rightarrow \Delta_H(\bar{y}, y) = 0.2$

Often used for graph labeling tasks, *e.g.* image segmentation, natural language processing, ...

Example 3: Squared error

If the individual variables y_i are numeric, *e.g.* pixel intensities, object locations, etc.

Sum of squared errors

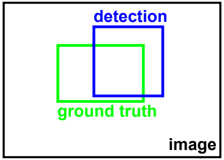
$$\Delta_Q(\bar{y}, y) = \frac{1}{m} \sum_{i=1}^{m} \|\bar{y}_i - y_i\|^2.$$

Used, *e.g.*, in stereo reconstruction, optical flow estimation, ...

Example 4: Task specific losses

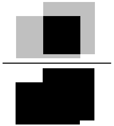
Object detection

- bounding boxes, or
- arbitrarily shaped regions



Intersection-over-union loss:

$$\Delta_{\text{IoU}}(y, \bar{y}) = 1 - \frac{\text{area}(y \cap \bar{y})}{\text{area}(y \cup \bar{y})} \qquad = 1 -$$



Used, *e.g.*, in PASCAL VOC challenges for object detection, because its scale-invariance.

Making Optimal Predictions

Given a structured distribution $p(x, y)$ or $p(y|x)$, what's the best y to predict?

Decision theory: pick y^* that causes minimal expected loss:

$$y^* = \operatorname*{argmin}_{\bar{y}\in\mathcal{Y}} \mathbb{E}_{y\sim p(y|x)}\{\Delta(y,\bar{y})\} = \operatorname*{argmin}_{\bar{y}\in\mathcal{Y}} \sum_{y\in\mathcal{Y}} \Delta(y,\bar{y})p(y|x)$$

For many loss functions not tractable to compute, but some exceptions:

- $\mathcal{R}_{\Delta_{0/1}}(y) = 1 - p(y)$, so $y^* = \operatorname{argmax}_y p(y)$ → use MAP prediction
- $\mathcal{R}_{\Delta_H}(y) = 1 - \sum_{i\in V} p(y_i)$, so $y^* = (y_1^*, \ldots, y_n^*)$ with $y_i^* = \operatorname*{argmax}_{k\in\mathcal{Y}_i} p(y_i = k)$
  → use marginal inference